Optimization Techniques for Big Data Analysis

Chapter 1. Introduction

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1. Introduction
    - Why take this course?
    - Basic concepts

2. Optimization problems in Machine Learning
    - ML setup
    - Most common optimization problems in ML

Motivation

Optimization is a supporting technology in many numerical computation-related research fields, such as machine learning, signal processing, industrial design, and operation research.

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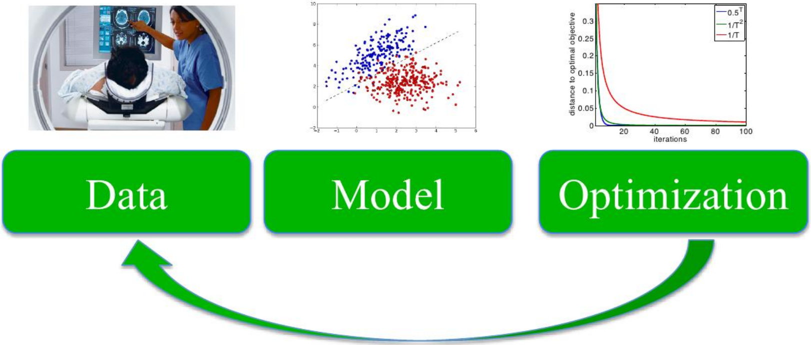

Data

Model

Optimization

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  It often appears as a relaxation of risk/error minimisation problems. The *learning* problem in many parametrized models involves Continuous Optimization.
- Discrete optimization:
  It occurs in inference problems in structured spaces, such as Feature selection, Data subset selection, Data summarization, Architecture search etc.

Continuous optimization in ML

- Supervised Learning: Logistic Regression, Least Square, Support Vector Machines, Deep Models.
- Unsupervised Learning: k-Means Clustering, Principal Component Analysis.
- Contextual bandits and Reinforcement learning: Soft-Max estimators and Policy Exponential Models.
- Recommender systems: Matrix Completion, Non-Negative Matrix Factorization, Collaborative Filtering.

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Countless ML libraries available implement all kinds of optimization algorithms (Tensorflow, PyTorch, Scipy, Sklearn, Vowpal Wabbit, ...)

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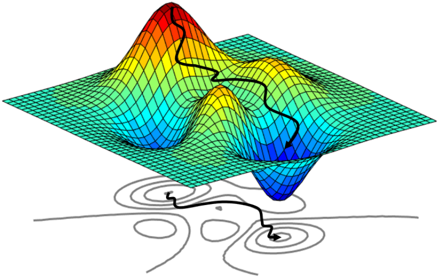
- Federated learning: Run the algorithm in different nodes without sharing any data among nodes.

Nomenclature

- For the most part through this course, we will use n as the number of training instances and d as the number of dimensions (features).
- Function: $f(\cdot)$
- Scalar: x; single-input function $f(x)$
- d-dimensional vector: $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$; $\mathbf{x}_i = [x_{1i}, x_{2i}, \cdots, x_{di}]$
- multi-input function $f(\mathbf{x})$; vector-valued function $\mathbf{f}(\mathbf{x})$
- Vector Space: $\mathcal{X}$
- Vector Norm: $||\mathbf{x}||_L$
- Inner product: Given two vectors $\mathbf{w}, \mathbf{x} \in \mathbb{R}^d$, define the inner product $\langle \mathbf{w}, \mathbf{x} \rangle = \sum_{j=1}^{d} \mathbf{w}_j \mathbf{x}_j = \mathbf{w}^T \mathbf{x}$
- Random variable X; Matrix: $\mathbf{X}$

Mathematical optimization



"The selection of the best element, with regard to some criterion, from some set of available alternatives" [1].

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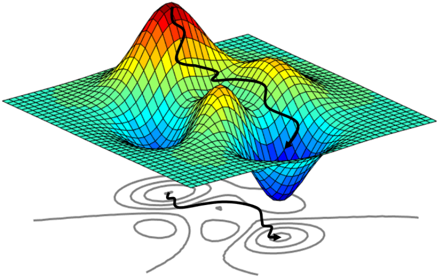


"The selection of the best element, with regard to some criterion, from some set of available alternatives" [1].

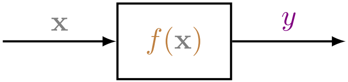
So typically we are given a problem like the following:

$$\hat{x} = \arg\max_{x} f(x) \;\; \text{s.t. } g(x) < a \tag{1}$$

- $f(\cdot)$ function subject to optimization.
- $x \in \mathcal{X}$ variables/parameters that need to be adjusted.
- $\mathcal{X}$ is the search space. $\hat{x}$ is the optimum.
- $g(\cdot)$ restrictions. $f|_{\mathcal{G}}$, where $\mathcal{G} \subseteq \mathcal{X}$; $\mathcal{G}$ is the feasible set.

ML set-up

A typical ML task can be seen like:



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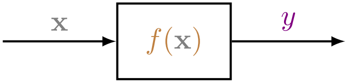

$\mathbf{x}$

$f(\mathbf{x})$

y

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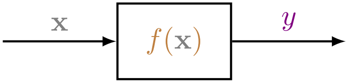

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Training $f_\theta(\cdot)$ corresponds to the optimization of $J(\theta)$!

Supervised Learning: Modelling

- Data: Given training examples $\{(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_n, y_n)\}$ where $\mathbf{x}_i \in \mathbb{R}^d$ is the feature vector and y_i is the label.
- Model: Denote the Model by $f_{\theta}(\mathbf{x})$ with θ being the parameters of the model. e.g. $f_\theta(\mathbf{x}) = \theta^T \mathbf{x}$
- Loss Functions: The loss function ℓ tries to measure the distance between $f_\theta(\mathbf{x}_i)$ and y_i.

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The generic problem denoted as the *expected loss function* can be expressed as:

$$\arg\min_{\theta} \mathcal{L}(\theta) = \mathbb{E}[\ell_\theta(\mathbf{x}, y)] + \lambda r(\theta) \tag{3}$$

where ℓ is the *instantaneous loss*, $r(\cdot)$ is the *regularizer* and λ a regularization factor.

Empirical loss functions

In most of this course' cases, we are dealing with a supervised learning problem, so given a dataset $\mathcal{D} = \{(\mathbf{x}_i, y_i)_{i=1}^n\}$ the loss function takes the form:

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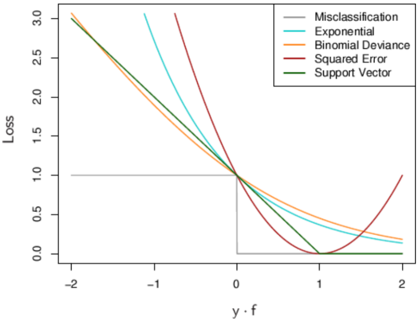
Examples of ℓ:

- Logistic loss: $\log(1 + \exp(-y_i f_\theta(\mathbf{x}_i)))$
- Hinge Loss: $\max\{0, 1 - y_i f_\theta(\mathbf{x}_i)\}$
- Absolute Error: $|f_\theta(\mathbf{x}_i) - y_i|$
- Least Squares: $(f_\theta(\mathbf{x}_i) - y_i)^2$



Instantaneous loss function

A common simplification approximates the original function by its instantaneous value:

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- When minimizing it, we will have to consider the gradient noise that will affect convergence properties.
- This simplification makes the algorithm very attractive in big data applications, both because of hardware requirements and for distributed settings.

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- High dimensional data: Because of the dimensionality of the parameter spaces, we can't afford second-order optimization methods. First-order or simplified second-order methods are requested.
- Distributed processing is highly desirable.

Acknowledgments

I would like to acknowledge several sources I have used to create slides

- Rishabh Iyer's course at University of Texas, DA
  `https://github.com/rishabhk108/OptimizationML`
- Martin Jaggi & Nicolas Flammarion's course at EPFL
  `https://github.com/epfml/OptML_course`

Questions?

[1] Melvyn W Jeter. *Mathematical programming: an introduction to optimization.* Routledge, 2018.

Thank You

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